Integer Partitions

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In partial fulfillment of the requirements for the degree of:

Masters of Science in Teaching Mathematics

Portland State University

Department of Mathematics and Statistics

Summer 2022

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Abstract

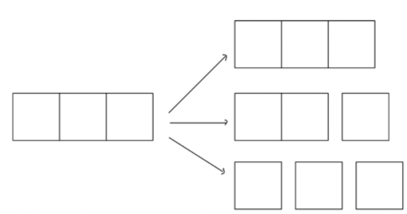
The theory of integer partitions has a long history, dating back to many of the great mathematicians, including Ramanujan, Euler, Legendre, and Hardy. Although the topic is very accessible to mathematics students of all ages, and although the motivating questions involve properties of integer arithmetic, many of the techniques applied to this study are often reserved for courses in advanced mathematics, such as analysis or combinatorics. Indeed, most of the existing literature on the theory of integer partitions is aimed at an audience of professionals in mathematics. In this project, the goal is to explore some of the main, foundational results of the theory of integer partitions (as presented in the textbook of Andrews and Erickson [1]), and then to prepare an expository portion of the project that will include a summary of major results and techniques, along with examples and appropriate justification. The curriculum materials developed are appropriate for the level of students typically found in an undergraduate introductory course on discrete mathematics. The curriculum materials will include materials I developed for this project, complete with support materials for any teacher wishing to implement these materials with their own students.

Introduction to the Mathematical Exploration

Many topics in combinatorics feel immediately accessible, compared to those in other areas of mathematics; the ideas of permutations and combinations have natural applications to real-life activities and scenarios: cards and dice, the number of toppings one might order on a pizza, or how many ways there are to distribute Halloween candy to a group of eager children, just to list a few examples. One of the great appeals of the subject is how simple it can be to think up situations where combinatorial questions might be asked, yet at the same time, the task of answering these questions can vary dramatically in difficulty. Combinatorial problems can be complex and subtle in their phrasing; variations on the fine print can often take the subsequent math in different directions and can have answers far less intuitive than the questions would initially imply.

My recollection of studying integer partitions for the first time is one of particular excitement: it was a topic that was clearly rich with difficult questions, but also distinctly *tangible*; by its nature, the analysis of integer partitions focuses on things you can touch and see – playing with natural numbers, like children's blocks, arranged and piled together in different ways. In our initial exposure to the topic during an undergraduate combinatorics course, we considered the question of *partition numbers:* given some natural number n, in how many different distinct ways could you split this n into (potentially) smaller nonempty piles (or parts)?

For example, splitting $n = 3$ into parts yields three possibilities:



These partitions correspond to the arithmetic facts that 3 = 3, and 3 = 2+1, and 3 = 1 + 1 + 1. This kind of problem seemed real and tactile, which, for me, made it especially fun.

However, we all quickly discovered that even such foundational questions as this were not simple; as one moved beyond the first three or four natural numbers, the partition numbers themselves threatened to spiral larger and larger, beyond the realm of simple case breakdowns. Less-than-systematic approaches to tracking all the different splittings quickly became untenable. However, even when the answers became harder to pin down, the process never lost its sense of excitement – the mysteries of integer partitions felt *important* somehow, as if their mysteries reached further than we knew at the time. This turned out to be true: integer partitions, I now know, have connections to diverse topics such as number theory, generating functions, the Fibonacci numbers, Gaussian polynomials, and plane partitions, among others.

As I approached designing this curriculum, I realized that I'd have to narrow my focus to touching on just a few of these areas; a fuller, more comprehensive survey of integer partitions could easily fill a semester-long course, if not more.

Since this curriculum will be, for most students, their first exposure to the field of integer partitions, I've kept it fairly rooted in the basics – the activities all rest upon geometric representations of integer partitions, often using *Ferrers diagrams*, a particular way of representing partitions that makes certain identities and properties easier to grasp. Each activity will contain some suggestions for extensions that can help students dive more deeply into the topic, but there are still many more possible activities and applications not covered here that would be rich material for future curricula.

Integer Partitions

Definition

Let n be a natural number, and let $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$ be natural numbers such that $\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_k = n$, and $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_k$. (We adhere to this weakly decreasing ordering throughout – more on this below). We say that $\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_k$ is a *partition* of n; in other words, n has been written as a **sum of positive integers**.

One way to represent partitions of integers is by using *dots*. In each of the partitions below, each dot represents a value of *1* (thus, for any partition of n, there will always be n total dots). Each *row* represents a pile (or part) of the partition.

Example: $n = 4$. There are five ways to partition the number 4, shown below:

We read the first partition as 1 + 1 + 1 + 1, the second as 2 + 1 + 1, the third as 3 + 1, and so on.

It should be mentioned that when studying integer partitions, we assume that order does not matter; thus, for example, there is only one partition of 4 into 2 + 1 + 1, since other orders, such as 2 + 1 + 1 or 1 + 2 + 1, etc., would not be weakly decreasing. Indeed, if we assume that all partitions are written in weakly decreasing order, then this should not present any confusion for the student, especially once they understand how to interpret Ferrers diagrams, which are formally introduced later.

For any natural number *n*, we say that the *partition number* is the total number of distinct ways that *n* may be partitioned.

The Partition Function p(n)

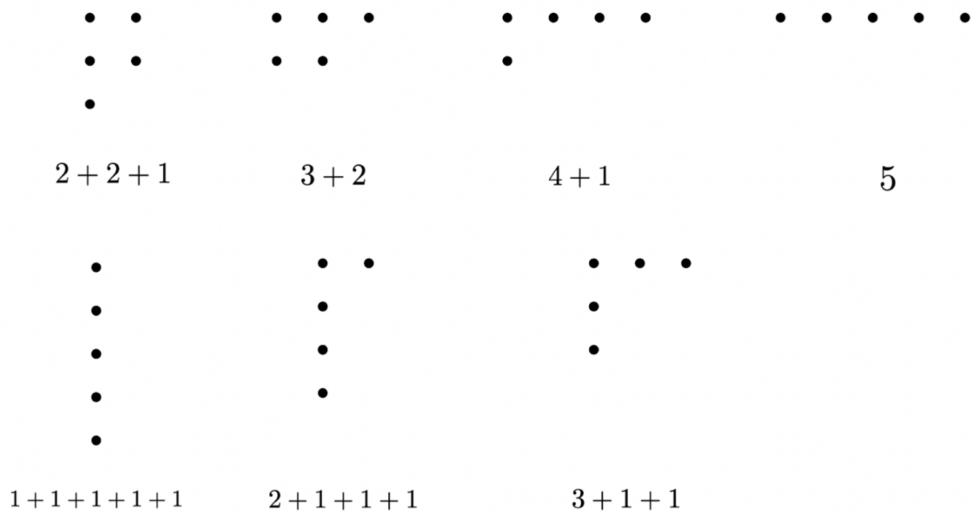
The example above, showing that $n = 4$ can be partitioned in five distinct ways, is an example of the *partition function*:

Definition: For any natural number *n*, the **partition function** *p(n)* is the number of distinct ways of representing *n* as a weakly decreasing sum of positive integers.

Looking at our previous example through this new definition, we would say that $p(4) = 5$; in other words, the partition *number* is the output of the partition function.

If we look at another example with a larger *n*, we see how quickly calculating the output of this function can become unwieldy.

Example: $n = 5$. There are seven ways to partition the number 5, shown below:



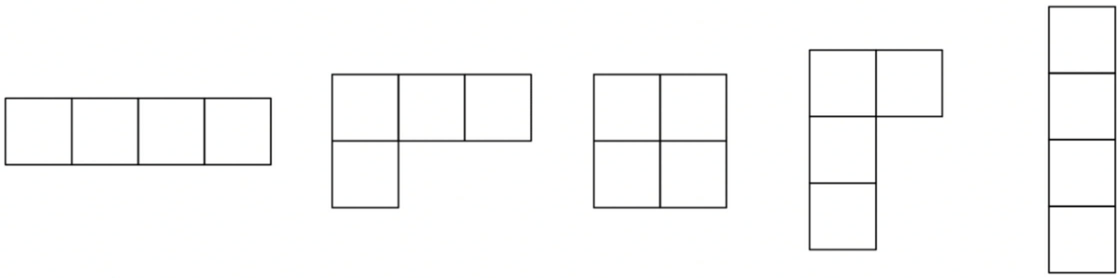
The following table gives the partition numbers up to $n = 20$:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
p(n)	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	490	627

Looking at such a table, students may struggle to discover the formula for such a function (technically, certain formulas or expressions *do* exist, but they are quite technical, far from intuitive, and deriving them is beyond the scope of this curriculum). Even a student with some familiarity with number theory or sequences and series might struggle to identify a pattern – for example, going by the above table for $n \in [2, 6]$, the partition numbers seem to correspond to the prime numbers; however, this breaks down for $n \geq 7$. However, just playing with some low-level examples may be sufficiently intriguing to students to motivate them to study the matter further.

Ferrers diagrams

In the earlier examples, we visualized different partitions of certain integers using dots; these representations are one version of what are called *Ferrers diagrams* (or Ferrers graphs). The other way they are commonly represented is by the use of squares (or blocks). Here, again, are the partitions of $n = 4$, using squares instead of dots:



4 $\quad$ $3+1$ $\quad$ $2+2$ $\quad$ $2+1+1$ $\quad$ $1+1+1+1$

Each of these partitions is a Ferrers diagram; as with the dot representation, each row in the diagram represents one *part* of the partition; further, the rows of Ferrers diagrams are also always given in weakly decreasing order, top-to-bottom, like before. This alignment is consistent with our supposition that in integer partitions, the order the parts are written in does not matter, so we default to a weakly decreasing order to avoid confusion. We will follow this pattern whenever we deal with Ferrers diagrams.

We note: The number of rows in any Ferrers diagram is equivalent to the number of parts n is broken into; similarly, the length of the longest row (or rows) in the diagram represents the largest part(s) in the partition. This example also illustrates that for any $n > 1$, there are always, at minimum, two trivial partitions of n: One into one part of size n (the diagram on the left) and into n parts of size 1 (the diagram on the right).

Partition Identities

Conditions on Partitions

The original partition function, complex as it is, is merely a starting point for our exploration. Where the world of integer partitions truly crosses over into the weird and wondrous is with the incorporation of certain *conditions* on the partition numbers: For instance, rather than just a general counting of all partitions, what if we restricted ourselves to counting only the number of ways to partition some natural number n so that no two parts have the same size? We might write this using the notation $\mathrm{p}(n \mid \text{distinct parts})$.

How would this constraint affect the outcomes? There are some immediate consequences: for even n, for example, the partition of n into two equal parts of size $\frac{n}{2}$ would immediately be disqualified; however, this is far from the only sort of way we could get repeated parts. We will discuss a more precise method for counting such conditional partitions in the next section.

Another example: How many partitions are there of some given n such that there are only an *even number of odd parts*? In this corner of the mathematical universe, the notation is again necessarily descriptive: $\mathrm{p}(n \mid \text{even number of odd parts})$.

This is to say: Count only the partitions in which the *number* of odd parts is even. How can we think about this? How do we begin to find a systematic way to count such integer partitions, where each new condition seems to take it further and further from the (comparatively) pleasant realm of the original partition function?

Bijective Proofs of Partition Identities

In my research for this project, I discovered that one of the most common ways to count specific conditional partition numbers is by using *bijective proofs* – similar to other topics in math whereby identifying a bijection between two sets guarantees an equivalence in their cardinality. Applied to integer partitions, this has two immediate benefits. First, by linking what might be one very complicated or difficult conditional partition to another (which may be easier think about/play with), we can more confidently determine the partition number of the more complicated one (assuming we are able to count the simpler one successfully). Second, using this method this reveals many intriguing and oftentimes surprising relationships between different types of groups of numbers. This second benefit is the real jewel -- while it can often be difficult to find an exact number for certain partitions, this is often less meaningful than finding unexpected equivalences between seemingly unrelated classes of partitions.

For many students, the process of finding bijective relationships between sets can be a challenge; one of the nice things about considering them using integer partitions is the inherently visual/tangible aspect of the subject. In the section below, one of the most well-known integer partition bijections is given. However, in a classroom environment, it may be worthwhile to allow students time to play with trying to find bijections between different partitions on their own, without any scaffolding (prior even to introducing them to the technique described below). By allowing themselves the space to play with small examples, students can begin to develop their instincts for relationships between different types of partitions, even if they do not fully understand *why* they exist.

Euler's Identity

To begin to explore bijections on integer partitions, let's examine one of the most famous: Euler's Identity.

$$p(n \mid \textit{odd parts}) = p(n \mid \textit{distinct parts}) \text{ for } n \geq 1$$

The identity makes the claim: for any $n \geq 1$, the number of ways to partition this n into *odd* parts is equal to the number of ways to partition it into *distinct* parts (no two parts the same size).

Let us start by considering the gentle case of $n = 5$.

If we try to systematically list all partitions of 5 into *odd* parts, we may start by partitioning 5 entirely, then eliminating all partitions with even parts. Suppose we list *all* partitions of $n = 5$ by *number of parts*, low to high:

5

4 + 1

3 + 2

3 + 1 + 1

2 + 2 + 1

2 + 1 + 1 + 1

1 + 1 + 1 + 1 + 1

So p (5) = 7. Of these, 4 + 1, 3 + 2, 2 + 2 + 1, and 2 + 1 + 1 + 1 are disqualified (since they contain even parts). So, by elimination,

$$p(\ 5 \mid \textit{odd parts}\ ) \ = 3.$$

So, according to the identity, there should be exactly three partitions of 5 into *distinct* parts as well. Looking at the initial list, we see that they are the partitions 5, 4 + 1, and 3 + 2.

This example, while manageable, is not entirely convincing; why would these two be equal in general?

The proof hinges upon the idea of *merging* and *splitting*. The idea is this: In order to pair each partition of odd parts with exactly one partition into distinct parts, we need only concern ourselves with any instances of *repetition* – i.e., partitions where certain part sizes occur more than once. Wherever we find such repeated parts, we merge them, creating a single part twice as large, one pair at a time. We repeat this process until no more repeated pairs remain.

So, let us again consider our three partitions into odd parts:

$$\{5,\ 3+1+1,\ 1+1+1+1+1\}.$$

In the case of the first partition, the 5 is already distinct; so, we leave it alone (trivially, this also results in a partition into distinct parts).

In the second case, we have two 1's; thus, according to the technique, we merge these:

$$3+1+1 \mapsto 3+2$$

Since there are now no more repeated parts, the process terminates.

Finally, for the partition into all parts of size 1, the merging occurs in stages:

$$\begin{aligned} 1+1+1+1+1 &\mapsto 2+2+1 \\ &\mapsto 4+1 \end{aligned}$$

It is worth mentioning here (to avoid any confusion that students may have): there are technically *five* repeated parts, but, the process merges only *two at a time*; thus, order not really mattering, one pair of ones would merge, then another pair; the remaining one would *not* be absorbed at that point, since it would become distinct. However, by these initial mergings, we then have repeated parts of size 2, so they *also* merge, giving us our final (and distinct) partition of 4 + 1.

By design, this process necessarily terminates with all parts being distinct.

To complete the picture, however, we need to explore the other direction: sending each distinct partition to exactly one partition with all parts odd.

To go from odd to distinct, we used the technique of merging; to go back, we use the natural inverse of this idea: *splitting*. Given some distinct partition of n, we focus only on any *even* parts it contains; parts that are odd to begin with are left alone. Each even part is split into two equal, smaller parts (for instance, 6 would be split into two parts of size 3), one pair at a time. If each of these smaller parts is also even, it too is split into two parts of equal size, as many times as necessary for each until each of the two smaller parts is odd (this must always eventually terminate – since any natural number is divisible by two finitely many times). The result is a partition into odd parts.

On the one hand, this merging/splitting technique feels very intuitive, almost obvious (however, without knowing what to look for when seeking a bijection between the two types of partitions, it may not be as obvious, especially to students).

Another thing worth commenting on is that, while the merging/splitting approach seems to do the trick of turning any partition into odd parts into one in distinct parts, or vice-versa, we might still be skeptical that this is truly a bijection. Might not there be two different partitions into odd parts which, when they undergo the splitting process, land at an identical partition into distinct parts? Similarly, could we not conceive of two different partitions into distinct parts that then were split into the same partition into odd parts? How could we be sure, when this entire process is based on the idea that we are not actually counting the number of partitions of each type (indeed, it may be very labor-intensive to do so)? How could we know?

As is often the case with integer partitions, a formal proof is neither obvious nor appropriate for students just learning about the topic (many of them involve more advanced applications of proof by induction). For our purposes, an informal description will suffice.

Suppose that M is a partition of some n into distinct parts. If every part of M is already odd, then there is nothing to do; so, let us assume that there exists at least one even part in M. As described, the splitting process will take this part and break it into two smaller parts of the same size. Are these parts both odd? If so, nothing more is done; if they are still even, then the process splits each of them again, until only odd parts remain. If there is more than one even part in M, this process continues until all have been processed.

How can we feel sure that this process guarantees a *unique* odd partition when all is said and done? One key fact here is that each splitting *does not affect any other*; in other words, the order of the splitting was arbitrary. Since every even part in M is eventually split into some collection of odd parts (and this end result is uniquely determined for each such part), the *total* partition into odd parts will also be unique.

A similar argument can be made in the other direction. Suppose that M' is a partition of n into odd parts. As before, if there are no repeated part sizes, there is nothing to merge; the partition is already distinct.

Suppose, then, that there exists at least one pair of nondistinct parts. By our method, these parts are merged into a single part, twice as large. If the two original parts were the only same-size pair, we are done; if another exists in the original partition, it too is merged, and on and on.

As with the splitting, the *order* of each merging of pairs does not affect any other. However, there is one extra point worth mentioning here: It was essential to this technique that the starting set of parts be *odd*; if we allowed for even-sized parts, it is conceivable that two different partitions could be drawn up that would "merge into each other" by this process. For instance:

$$2 + 2 + 4 + 1 + 1 + 1 \qquad \text{and} \qquad 4 + 4 + 1 + 1 + 1$$

Our technique would merge the 2's in the first partition, thus turning it into the second partition; from there, both would follow the same road.

Why is this avoided when we restrict ourselves to odd parts only? It has to do with the fact that, given any two repeated odd parts, they merge to form an even part – and because, by construction, integer partitions contain a finite number of parts, no matter how many different odd parts we begin with, the initial merging of all repeated odd parts *must* result in a uniquely determined set of even parts, repeated or otherwise, which will then continue to merge by this process until all parts are distinct. Our above counterexample showed that if we allow in even parts, it can become unclear *where* in the process we are, i.e., if one partition is not really another in disguise.

Conjugate Partitions

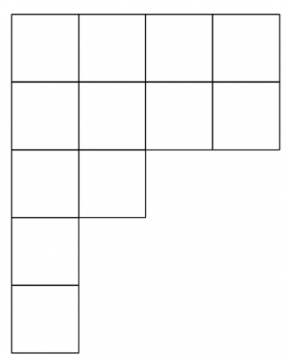
Introduction

For many students, Euler's identity provides a solid introduction to the idea of bijections on integer partitions, due to the simplicity of the technique.

Another operation on integer partitions which both sheds light on interesting and unlikely bijections and is intuitively straightforward is that of *conjugation*.

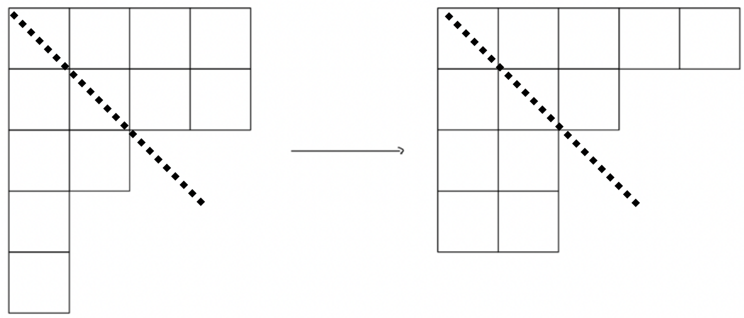
To explore this idea, we return to the concept of Ferrers diagrams introduced earlier.

Consider the Ferrers diagram below.



We may read this as 4 + 4 + 2 + 1 + 1, a partition of n = 12.

Referring to this diagram of this partition, we define *conjugation* as a transformation which essentially transposes the diagram along the main diagonal, swapping the horizontal with the vertical:



Visually speaking, if we squint, we see that the long left-side column of the first partition is now the *top row* of the second, and so on.

The original partition was 4 + 4 + 2 + 1 + 1; looking at the second partition, we read it as 5 + 3 + 2 + 2; a completely different partition of the same number of blocks! Taken together, we call these two *conjugate partitions*.

When students play with the idea of conjugate partitions, they will notice a few immediate consequences. One is that under conjugation, the *number* of rows of one partition becomes the *longest* row of the other. This is given by the following theorem:

$$p(\, n \mid m \text{ parts}) = p(\, n \mid \text{greatest part is } m)$$

There is a corollary to this theorem:

$$p(\, n \mid \leq m \text{ parts}) = p(\, n \mid \text{all parts } \leq m)$$

A sketch of a proof is given below.

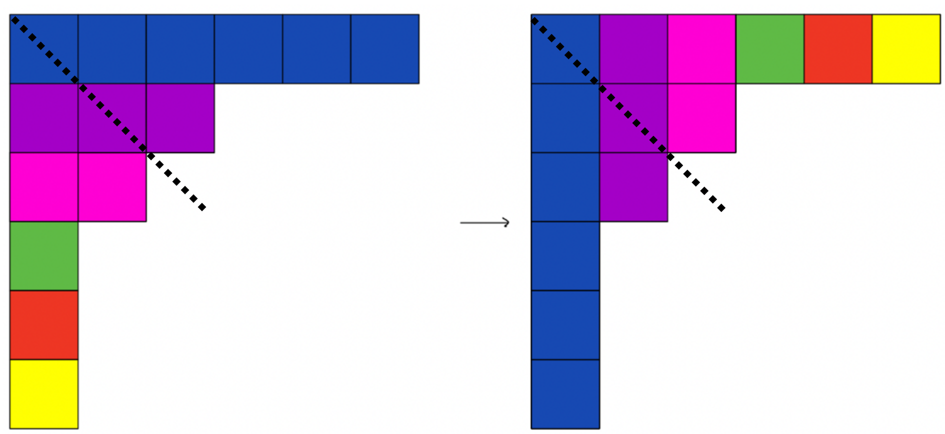
Suppose that some n is partitioned into m parts. In the language of Ferrers diagrams, this is equivalent to the partition having m rows. Thus, under conjugation, we may imagine the left-side column of the original diagram becoming the largest part of the conjugate partition (this is the essence of the first theorem above); the theorem holds whether or not this greatest part is unique in the conjugate partition.

The corollary proceeds in an inductive manner. If n is partitioned into m parts, then the proof is done. If, however, n is partitioned into *fewer* than m parts, then there still must exist at least one largest part of this partition (call it m_1). Supposing that $m_1 < m$ means that there are fewer parts in this partition than our original one. Under conjugation, this m_1th part (or, row in the Ferrers diagram) becomes the largest part of the conjugation partition, of size m_1. And since m_1 was an arbitrarily chosen part (less than or equal to the m_{th} part), we may generalize this for all parts less than or equal to the original largest part.

Self-Conjugate Partitions

Definition: A partition is said to be *self-conjugate* if it is equal to its own conjugate.

A visual example will help to demonstrate this concept.



One interesting identity that utilizes conjugation is

$$p(n \mid \text{self-conjugate}) = p(n \mid \text{distinct odd parts})$$

Verifying this identity using Ferrers diagrams may be a useful introductory exercise for students. We will use conjugation to explore this, and more interesting and unintuitive bijections, in a later activity.

Miscellaneous Extensions

An Upper Bound on p(n)

One intriguing question that comes up when you start to study integer partitions is, *just how fast do the partition numbers p(n)* grow?

This is a complicated question, and beyond the scope of this paper; however, there are some interesting observations to be made. Let us first formally establish (beyond an intuitive sense) that p(n) is an *increasing* function.

We wish to show that

$$p(n) > p(n-1) \textit{ for all } n \geq 2$$

An informal proof is given below.

Suppose $n = 3$. The three partitions of n are shown below:

1 + 1 + 1   2 + 1   3

To each of these partitions, we may add a single dot to a new bottom row, creating certain partitions of $n = 4$:

It follows that given the partitions of *n*, we can form new select partitions of n + 1 by adding a single dot in a new row to each specific partition (as we did above). This brief example gives the following identity:

$$p(n-1) = p(n \mid \text{at least one 1-part})$$

However, what about partitions of 4 that do not have a 1-part (that is, a row with one dot)? We know, trivially, that included in the partition of any natural number is the number itself (for instance, 4 into one part of size 4). There is also the partition of 4 into 2 + 2. These, combined with our above three partitions, yield the full partition set of 4, as we saw earlier:

Since there is – at minimum – always at least one partition of *n* (assuming *n* ≥ 2) *without* a part of size one (and often more than one), we arrive at the following identity:

$$p(n) = p(n-1) + p(n \mid \text{no 1-part}) > p(n-1)$$

Thus, p(n) is an increasing function.

In putting together this curriculum, I also learned that the upper bound for the partition function p(n) turns out to be the (*n* + 1)st *Fibonacci number* F_{n+1}! This is awesome, not at all obvious, and, as with many other connections between integer partitions and other topics in mathematics, worthy of further investigation in other curricula.

The *Composition* of an Integer

There is one more interesting connection between integer partitions and the Fibonacci numbers worth mentioning here.

Let us define a *composition* of an integer to be a partition where the order of the parts *does* matter (this represents a break from our usual way of considering integer partitions). For example, suppose we consider the compositions of integers into parts of size 1 or 2 only. It will be instructive to list out some of these compositions:

n = 1: 1

n = 2: 2, 1 + 1

n = 3: 2 + 1, 1 + 2, 1 + 1 + 1

n = 4: 2 + 2, 2 + 1 + 1, 1 + 2 + 1, 1 + 1 + 2, 1 + 1 + 1 + 1

n = 5: 2 + 2 + 1, 2 + 1 + 2, 1 + 2 + 2, 2 + 1 + 1 + 1, 1 + 2 + 1 + 1, 1 + 1 + 2 + 1, 1 + 1 + 1 + 2, 1 + 1 + 1 + 1 + 1

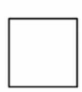
n = 6: 2 + 2 + 2, 2 + 2 + 1 + 1, 2 + 1 + 2 + 1, 2 + 1 + 1 + 2, 1 + 2 + 1 + 2, 1 + 2 + 2 + 1, 1 + 1 + 2 + 2, 2 + 1 + 1 + 1 + 1, 1 + 2 + 1 + 1 + 1, 1 + 1 + 2 + 1 + 1, 1 + 1 + 1 + 2 + 1, 1 + 1 + 1 + 1 + 2, 1 + 1 + 1 + 1 + 1 + 1

Let us denote the number of each of these integers into parts of size one or two as the *composition number* C(n). From the above list, we get the following table:

n	1	2	3	4	5	6
C(n)	1	2	3	5	8	13

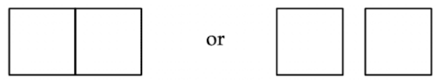
Considering this table, a pattern starts to appear: For each n, C(n) seems to be equal to F_{n+1}, the $(n + 1)^{\text{th}}$ Fibonacci number. This turns out to be true for all n – a sketch of a proof is given below. We will use horizontal rows of squares to represent each number n.

Starting with $n = 1$, we trivially observe that there is only one way to partition this number into parts of size 1 or 2:



So C(1) = 1, which is equal to F_2.

For $n = 2$, we now have the possibility of either leaving it as one part of size 2, or breaking it into two parts of size 1:



So C(2) = 2, which is equal to F_3.

From here, rather than continuing to list them out, we consider an inductive approach. For all $n \geq 3$, we first ask, does the rightmost square (the 'end' square, so to speak) in the row belong to a 2-part, or a 1-part?

If the end square belongs to a 2-part, it leaves $(n - 2)$ squares in the row to assign to either a 2-part or a 1-part. If the end square belongs to a 1-part, it leaves $(n - 1)$ squares in the row to assign to either a 2-part or a 1-part.

Having established $C(n) = F_{n+1}$ for the two base cases prior to $n = 3$, we already know how to count both of these cases; thus, for each increase of n, the total number of ways to partition it into parts of size 2 or 1 is *recursively defined*, with each new composition number being equal to the sum of the prior two.

Thus,

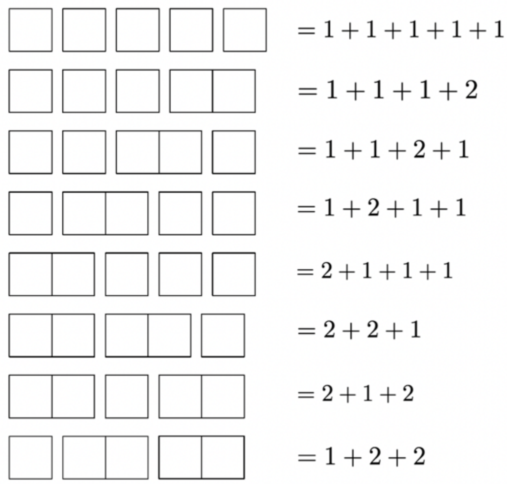
$$
\begin{aligned}
C(n) &= F_{(n-1)+1} + F_{(n-2)+1} \\
&= F_n + F_{n-1} \\
&= F_{n+1}
\end{aligned}
$$

$\forall n \in \mathbb{N}$.

Connections to the Arithmetic Triangle

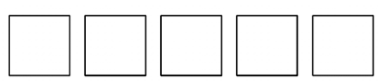
This proof regarding the Fibonacci numbers, while quite interesting, may be a stretch to integrate into a curriculum on integer partitions. However, there is another interesting combinatorial connection to be made from this particular brand of composition numbers, which may help form a bridge between the composition numbers and the Fibonacci numbers: the *Arithmetic Triangle*.

Let us reconsider the composition of $n = 5$ into ordered parts of size 1 or 2 (C(5)):



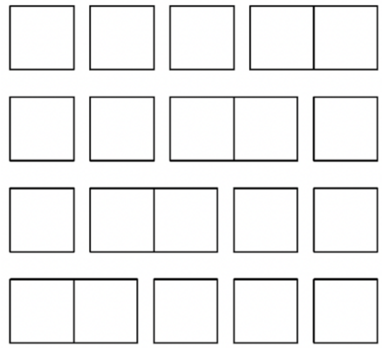
The method here is not too dissimilar from the inductive Fibonacci approach; here, however, we use a *summative* approach – we start from *no* parts of size 2, count them all, then count all distinct composition with *one* part of size 2, two parts of size 2, and so on.

If we start with the composition into all parts of size 1 (i.e., zero parts of size 2), then, trivially, there is only one such composition:



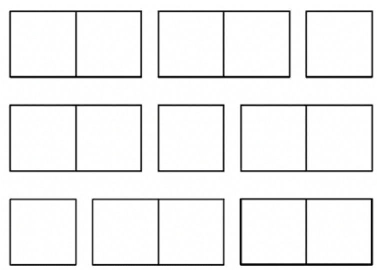
We may say that there are $\binom{5}{0} = 1$ such composition (where we are "choosing" the number of 2-parts).

We then consider the number of compositions with *one* part of size 2:



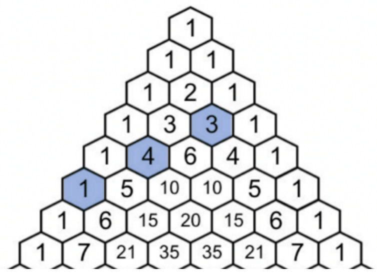
Here, because we have one 2-part, the *total* number of parts has gone down by one, so we are looking at $\binom{4}{1} = 4$ compositions of this type.

We next consider compositions with *two* 2-parts:



With two 2-parts, we've reduced the number of 1-parts to one, for a total of three parts. So, what we have here is $\binom{3}{2} = 3$ distinct compositions of this type.

There are no more parts of size two we can pull out, so we have exhausted all possibilities. So, for $n = 5$, we have $\binom{5}{0} + \binom{4}{1} + \binom{3}{2} = 8$ total ordered partitions (or compositions), which, as we noted before, is equal to F_6. But, since we've approached this in a combinatorial manner, we now consider this result in light of the Arithmetic Triangle:



Our composition number C(5) is *embedded* in the Arithmetic Triangle as the sum of entries along a sort of diagonal line, starting at row 5, column 0: 1 + 4 + 3 = 8.

We need not list the visuals of larger n to see if this pattern keeps up. Considering $n = 6$, let k be the number of 2-parts in our ordered partitions:

If $k = 0$: There is $\binom{6}{0} = 1$ way to partition six.

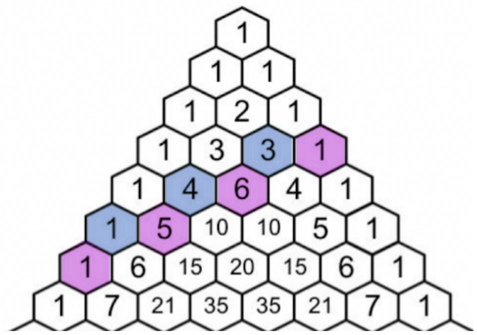
If $k = 1$: There are $\binom{5}{1} = 5$ ways to partition six.

If $k = 2$: There are $\binom{4}{2} = 6$ ways to partition six.

Unlike when we considered $n = 5$, we can go one step further here, letting k = 3:

If $k = 3$: There is $\binom{3}{3} = 1$ way to partition six.

This gives us 1 + 5 + 6 + 1 = 13 total partitions of six into ordered parts of size 1 or 2, which is equivalent to the 7th Fibonacci number. Looking again at the Arithmetic Triangle, we see:



So once again, we are finding our Fibonacci / composition numbers represented as the sum of entries along this diagonal!

Looking at the Arithmetic Triangle representations of C(5) and C(6), we note something interesting: If *n* is odd, then there can be no partition into all parts of size two, but if *n* is even, there can (the addition of the final 1 shown in our second composition above). From this, we generalize to an informal identity:

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} = C(n)$$

$$= F_{n+1}$$

Summing over *k* up to the floor function value of $\frac{n}{2}$ accounts for the variation between even and odd integers, and, as we noted in our examples, each time *k* goes up by one, the total number of parts *n decreases* by one. For even values of *n*, this function allows for the case of all parts of size 2; for odd values; the function stops one short of this.

The Arithmetic Triangle, which may be familiar to undergraduate-level students in a combinatorics class, is already rich with mathematical applications (especially from a counting perspective), so it is pretty awesome to see it also connect to the Fibonacci numbers and the integer partitions in this way!

Activity 1: Long Rectangles

Introduction/Notes on the Activity

In my reading for this project, I came across a problem which illustrated integer partitions using the concept of "long rectangles." At first, it didn't really catch my attention as a problem with too much to offer students in terms of insight into integer partitions, at least as I'd been thinking of them up to that point; from the perspective of Ferrers diagrams, these long rectangles were fairly unremarkable. However, the more I delved into it / played with examples, the more interesting a problem it seemed in its own right, with several unexpected connections to other mathematical concepts.

In this activity, students are given a fairly trivial Ferrers diagram: one row of blocks (with the lengths varying group to group). For each starting row, students then add on a sort of "long L" atop the initial row: for instance, a 1 x 4 diagram becomes a 2 x 5 diagram, then a 3 x 6 diagram, and so on (as shown in the worksheet below). The key fact here from an integer partition standpoint is that the amount of increase *between* the added parts goes up consistently by 2 at each stage: So, 1 x 4 = 4, 2 x 5 = 4 + 6, 3 x 6 = 4 + 6 + 10, and so on. The initial aim of the activity is to have students play with this constructive technique, make notes of any patterns they see, and record their data, with the ultimate goal of trying to write a function to describe the total number of blocks as a *function* of the starting row length and number of expansion steps.

For the second part of the activity, students will be given certain value *s* (which represents the total number of blocks in some long rectangle) and asked if they can work from that value *backwards* to reconstruct a starting row, and how many expansion steps would be taken to arrive at it. The idea here is to challenge students to think about the idea of *preimages* of function outputs using this geometric approach, and to help them think about the function being either injective or surjective.

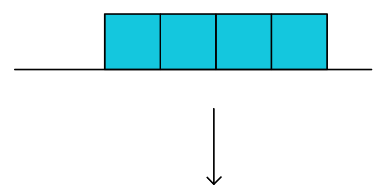
Worksheet (Day 1)

Long Rectangles Worksheet

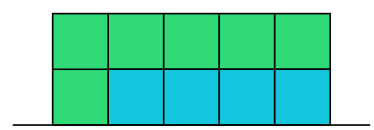
Instructions: In this activity, we're going to be building an ever-expanding rectangular structure (a math fortress, maybe?) using square blocks. The way it will go is this:

You begin with a single row of blocks, and you expand your structure one layer at a time. The way you expand it is to add an 'L' shape on top of the base row (as shown below). The result with each step will be a new, larger rectangle: what we are calling a **long rectangle**, meaning a rectangle where the width is always greater than or equal to the height.

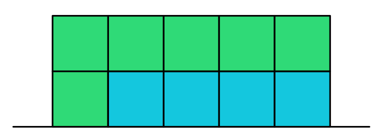
For example: Here is a starting row with 4 blocks (with a line below indicating "ground level"):



We then add an 'L' shape over this row:

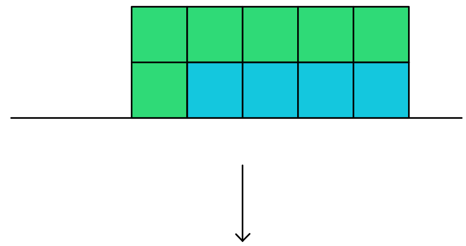


The result is an expanded long rectangle.

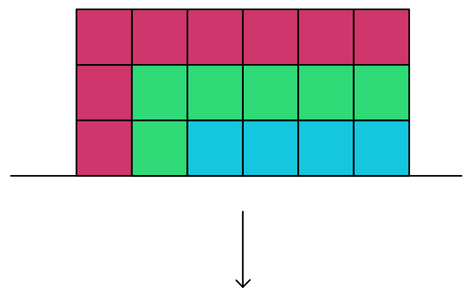


The next expansion would look like this:

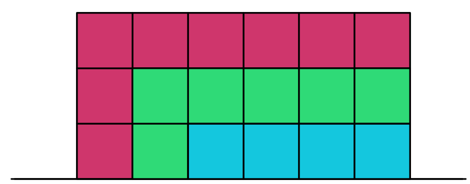
Starting with our previous rectangle…



We add a new 'L' shape over it,



and the result is a new rectangle.



Your first task is to take the starting row you've been given, and add layers to it. As you do, discuss amongst your group:

1. How is the total number of blocks growing at each step?
2. How might you track this?
3. What quantities seem relevant? Are there others that are less relevant?
4. What's a good way to represent your data?

Your starting row has _______ blocks (each group of students will be given a different starting number of blocks).

Use the given space to work in your group, to play around with the expansion, and note any patterns you see.

What did you notice? How did you track each step of the expansion? Did you notice any patterns?

Now, using the below table, fill in the data you collected as a group.

$r =$

(length of starting row)

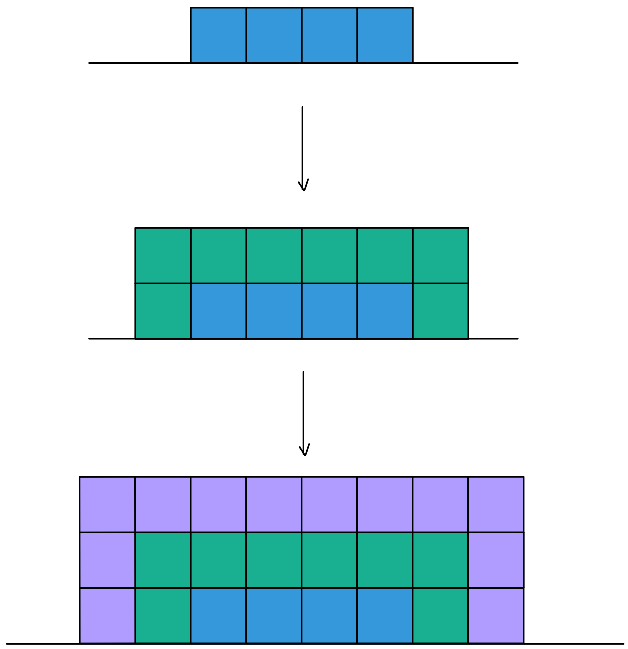
ℓ (layers)	1	2	3	4	5	6	7	8
s (sum of all the blocks)								

With your data captured in this way, your next task is to try and write a *function* for *s.*

What would such a function look like? What would be the input(s)? Work in your group and brainstorm ideas.

Bonus Activity

This activity is similar to the original, except instead of adding a 'long L' to construct new rectangles, we expand the structure on *both* sides, as shown below:



Using this new pattern of expansion, play the same game as before: record data for several steps of expansion, and then see if you can write a new function for *s* based upon this alternate approach.

What has changed? Does this change make sense? Why or why not?

Name:__________________

Homework

Between now and next class, play out a few more long rectangles, with different starting *r* than you had today. Can you find a function for *s* with these different starting rows, similar to what you did in class?

After playing through a few examples, here's the challenge: Can you find a function for *s* that works for *any* starting value of *r*? (Hint: Look at the functions you wrote for the long rectangles you created with the specific values for *r*, and see what they have in common)

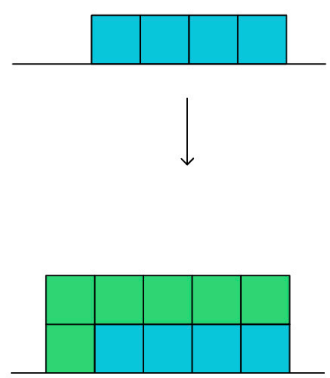
What would the *domain* of such a function be? What would the *range* be?

Here is another copy of the table we used in class, as well as a diagram of the expansion technique:

$r =$

(length of starting row)

ℓ (layers)	1	2	3	4	5	6	7	8
s (sum of all the blocks)								



Day 1 Activity Key

$$r = 4$$

(length of starting row)

ℓ (layers)	1	2	3	4	5	6	7	8
s (sum of all the blocks)	4	10	18	28	40	54	70	88

The function takes ***ordered pairs*** *(r, l)* as its inputs (in other words, the domain is the Cartesian product of the natural numbers). Given $r = 4$, the function would be:

$$s\left(l\right) = l \cdot \left(l + 3\right)$$

For the homework, students play with different starting values for *r* to detect patterns; ultimately, they should arrive at the following general functions for *s*:

$$s\left(r, l\right) = l \cdot \left(l + r - 1\right)$$

The domain is the Cartesian product of the natural numbers, and the range is the set of all natural numbers.

For the bonus question, students will notice that the difference between each added layer is going up by *4* instead of *2*.

Worksheet (Day 2)

In the last class, we talked about finding an explicit function for s in terms of l. For homework you considered a way to write a general function that worked for *any* $\boldsymbol{r}$ and $\boldsymbol{l}$. Here is one version of such a function:

$$s(r, l) = l(\ l + r - 1\ )$$

Now we're going to look at it from a different perspective.

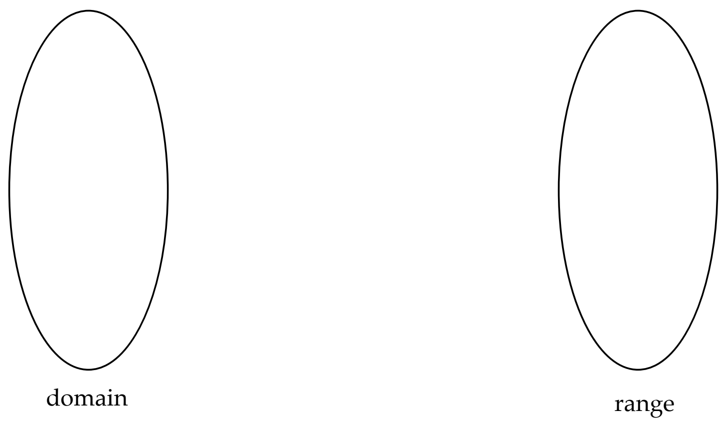
Suppose I tell you that after adding a certain number of layers, a long rectangle has *48 blocks* (in other words, $s = 48$). Working with your group, can you find an r and l that would generate this rectangle, using the process we practiced last class?

Are there any *s* where there are *no* possible long rectangles with that number of blocks? Why or why not?

Do you notice anything about the values of *s* which correspond to multiple long rectangles vs those which only correspond to one?

Now I'll give you another value for s (each group will have a different value). Since s is the output of the function we constructed, I want you to try to find as many *preimages* for s as you can, using the ovals below!

Your value of s is _______.



Having done this, do you think you could write an *inverse* function to describe this map? Why or why not?

Let's play with one more variation on this. I'd like you to work in your groups to construct new long rectangles using the method we practiced last class. But this time, *fix r = 1*.

$r = 1$

(length of starting row)

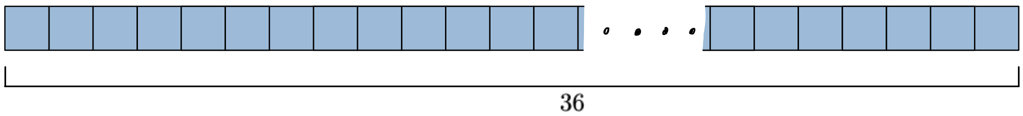
ℓ (layers)	1	2	3	4	5	6	7	8
s (sum of all the blocks)								

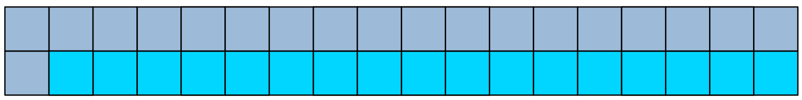
Construct the rectangles and use the table to track your data. What do you notice?

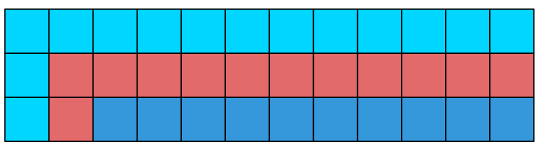
Connecting Back to Integer Partitions

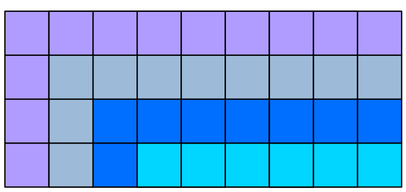
One thing we touched on at the beginning of this activity was tracking *how much* the total number of blocks in our rectangles in our examples were increasing step by step.

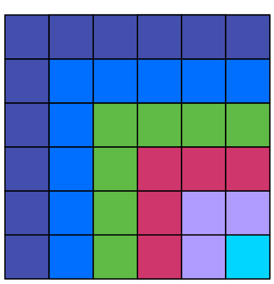
For $s = 36$, there 5 long rectangles with this number of blocks:











Describe these rectangles as the *sum* of the shaded regions.

(for instance, the first one would be: 36 = 36, the second one is 36 = 17 + 19, etc.)

Day 2 Worksheet Key

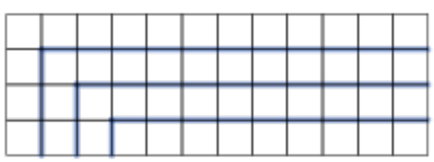
In the last class, we talked about finding an explicit function for s in terms of l. For homework you considered a way to write a general function that worked for *any* $\boldsymbol{r}$ and $\boldsymbol{l}$. Here is one version of such a function:

$$s(r, l) = l(\,l + r - 1\,)$$

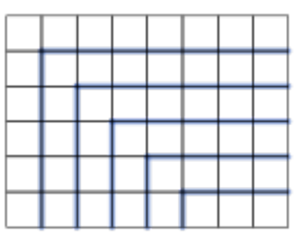
Now we're going to look at it from a different perspective.

Suppose I tell you that after adding a certain number of layers, a long rectangle has *48 blocks* (in other words, $s = 48$). Working with your group, can you find an r and l that would generate this rectangle, using the process we practiced last class?

Since 48 has numerous distinct factorizations, we will have several possible answers. A few are shown below:



$$s\,(9, 4) = 48$$



$$s\,(3, 6) = 48$$

$$s\,(23, 2) = 48$$

The key insight here is that *any* long rectangle is necessarily a factorization of s, so by starting from this fact, students may discover several answers.

Are there any s where there are *no* possible long rectangles with that number of blocks? Why or why not?

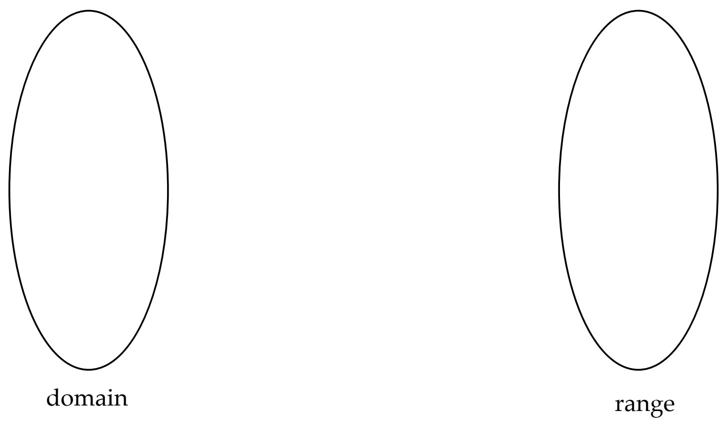
No. Since l may always be trivially set to 1, a long rectangle can be made for any natural number s simply by having an initial row of length s.

Do you notice anything about the values of s which correspond to multiple long rectangles vs those which only correspond to one?

After the above exercise, the idea here is that *composite* numbers correspond to values of s with more than one long rectangle, whereas prime numbers will correspond to only one (since they are only factorable in one way).

Now I'll give you another value for s (each group will have a different value). Since s is the output of the function we constructed, I want you to try to find as many *preimages* for s as you can, using the ovals below!

Your value of s is _______.



Having done this, do you think you could write an *inverse* function to describe this map? Why or why not?

In general, an inverse function is not possible, because (as the above work highlights), any s which is a composite number will, by definition, have *multiple* preimages; therefore, s itself is not a one-to-one function, and so is not invertible.

Let's play with one more variation on this. I'd like you to work in your groups to construct new long rectangles using the method we practiced last class. But this time, *fix $r = 1$*.

$r = 1$
(length of starting row)

ℓ (layers)	1	2	3	4	5	6	7	8
s (sum of all the blocks)								

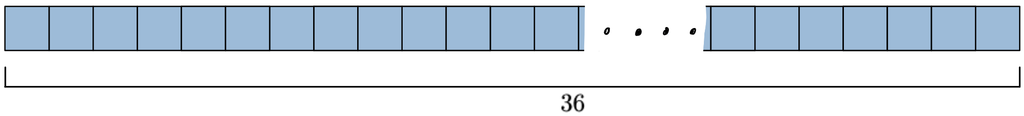
Construct the rectangles and use the table to track your data. What do you notice?

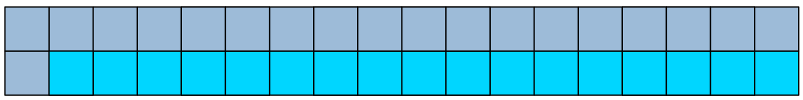
Letting $r = 1$ corresponds to s being the set of **perfect squares**.

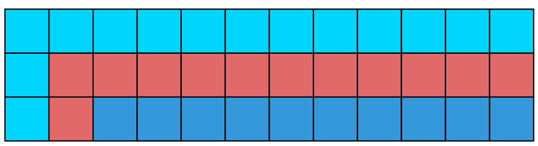
Connecting Back to Integer Partitions

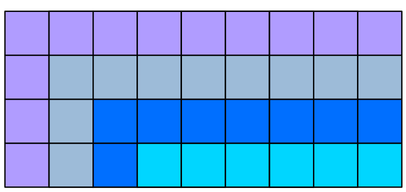
One thing we touched on at the beginning of this activity was tracking *how much* the total number of blocks in our rectangles in our examples were increasing step by step.

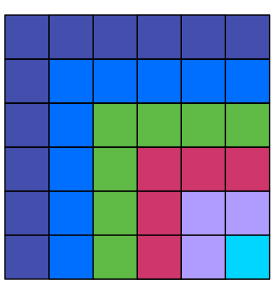
For s = 36, there 5 long rectangles with this number of blocks:











Describe these rectangles as the *sum* of the shaded regions.

(for instance, the first one would be: 36 = 36, the second one is 36 = 17 + 19, etc.)

The answers here would be:

36 = 36
36 = 17 + 19
36 = 10 + 12 + 14
36 = 6 + 8 + 10 + 12
36 = 1 + 3 + 5 + 7 + 9 + 11

Reflections on the Activity

Asking students to work with building their own rectangular structures step by step seemed like a fun task to assign, but I was concerned that it might be too big of a leap to ask them to go from this freeform, data-gathering stage to trying to write the function explicitly. At first, I considered giving students a pre-made table, complete with defined variables, to track their steps and the total number of blocks at each step, thinking that with this data in front of them, they could eventually make a conjecture for the function:

$r =$

(length of starting row)

ℓ (layers)	1	2	3	4	5	6	7	8
s (sum of all the blocks)								

However, thinking that some students might really struggle with this, I thought I should add in a row that looked at the rectangle at each stage in terms of its height and width; in other words, considering each s not as the sum of increasing parts, but as the product of factors. In my eagerness to help them make this connection, I modified the original table to include, for each stage in the expansion, the h x w factor-expression for the rectangle (here replacing l with h):

h (height)	1	2	3	4	5	6	7	8
$h \times w$ (height x width)								
s								

However, after thinking about it and running through some examples, I realized the dangers in this addition: It over-scaffolded the problem such that once students recognized the pattern of the factors in the table, they'd no longer feel the need to construct the rectangles using the (more instructive) partition method, and, more insidiously, it might short-circuit their process of thinking about constructing a function for *s* in *terms* of the variables of initial starting row and number of steps. By trying to lead them towards the connection between composite numbers and the non-injectivity of the function, I'd risked derailing them into a whole other area of mental consideration.

So, I scrapped the *h* x *w* row, and streamlined it back to just playing with the blocks, recording the *s* at each step, and letting them approach it more constructively. I do still feel that there is valuable number-theoretic insight to be gained from considering this activity from that perspective, but that this examination belongs firmly as a follow-up extension or addition, to be included or excluded at the teacher's discretion.

Given the time allotted for this activity, there is much that can be added as homework or as follow-up questions; even though the activity only lightly touches on integer partitions as a concept, there is an interesting line to be drawn between these partitions of sums and the different factorizations of the natural numbers.

Reflections on Student Work

For each day of the activity, I gave students copies of the worksheets and collected them at the end of class. Many groups of students picked up on the patterns quickly and anticipated the direction I was going; there was a good variety of methods used in capturing the data and deciding which quantities were relevant and which were not. Most groups hit upon the quadratic relationship at the heart of the long rectangle function early on. That said, there was an impressive variety of algebraic and sequential approaches to writing the explicit function for *s*.

One challenge that students faced was correctly identifying the *domain* of the function s(r, *l*), with many thinking that it was the set of natural numbers rather than the *Cartesian product* of the natural numbers (only a few groups correctly identified this). From what I could tell, this seemed to be more a matter of the amount of time they had than anything else; nearly every group correctly identified the function as multivariate, but perhaps just needed an extra nudge to connect this fact to the correct representation of the domain.

For the second day's activity, the students' work was similarly excellent; most groups hit upon the fact that the function *s* was *not* one-to-one due to several different values of *s* having multiple preimages. Some groups correctly identified the *prime numbers* as the only values of *s* for which there was a unique preimage; however, my feeling reading through their work on the second day was again that, given more time, all groups could have eventually identified this fact and arrived at a good grasp of why it was so.

Extensions

This activity seems to have only scratched the surface in terms of its connections to other mathematical topics. In the time I had with the class, we explored connections to multivariate bijective functions and the relationship between integer partitions and the prime and composite numbers. One cool extension of this activity is found in the next activity on recurrence relations; however, students would also benefit from creating a general table for *r* and *l* based upon the formula they came up with in this activity without making the pivot into recursive functions, as it provides many interesting patterns for study.

Activity 2: Recurrence Relations on Long Rectangles

Introduction/Notes on the Activity

The intention with this activity is to be a sort of "alternate universe" approach to the long rectangle problem of the previous activity, which put the main mathematical focus on thinking about bijective functions and the relationship between the prime and composite numbers and uniquely determined long rectangles.

The genesis of this activity came about when I was analyzing the general 8x8 (r,l) table that I'd originally envisioned as an extension in the previous activity. Looking at that table, it eventually presented itself as an opportunity for students to study the relationship between different long rectangles from the perspective of *recurrence relations*. The hope for this activity is that by giving students a geometric representation of a recursive identity, they might strengthen their understanding of recurrence relations in general, and also use it as an entry point into further study on geometric relationships, and more complex recurrence relations.

The activity will begin similarly to the first, with students being briefed on the construct and meaning of long rectangles. However, unlike the first activity, we then shift gears to presenting students with a given recurrence identity for any long rectangle with $s(r, l)$ blocks (where students have no *explicit* formula for this). By withholding the formal expression for $s(r, l)$ from the previous activity, students will instead get the opportunity to play around with the recurrence identity geometrically, and from there, fill in the 8x8 table for r and l themselves, based only on their brief exposure to long rectangle construction.

Once the table is complete, students will then get to *analyze* it, and attempt to conjecture a general formula for $s(r, l)$, similar to what was done in the first activity. Here, however, the intention will be to set students up to explore their understanding of *proving* recurrence relations by using induction to verify that the explicit function they have conjectured is in fact equal to the recurrence relation version.

Note: Depending on when this activity is presented, the process of proving the identity could present quite a challenge for many students without adequate scaffolding. The induction proof is more complex than what some students may be used to, so it may be either left as an optional part of the activity (since the main benefit is the geometric representation and pattern-finding of the 8x8 table), or, the activity may be appropriate for an intermediate-level combinatorics course, where students already have some experience with induction and proving recurrence relation identities.

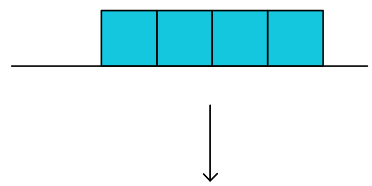
Worksheet

Long Rectangles & Recurrence Relations

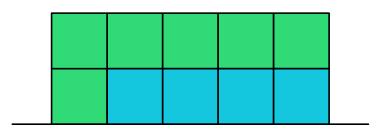
In this activity, we're going to study something called *long rectangles*, using square blocks.

To make a long rectangle, you begin with a single row of blocks, and then expand it one layer at a time. For example:

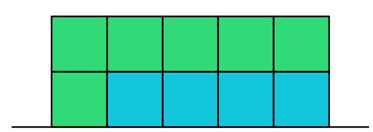
For example: Here is a starting row with 4 blocks
(with a line below indicating "ground level"):



We then add an 'L' shape over this row:

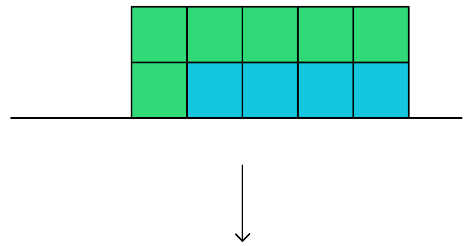


The result is an expanded
long rectangle.

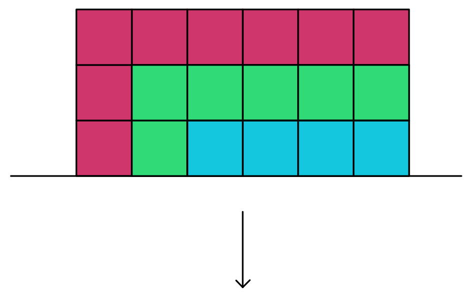


The next expansion would look like this:

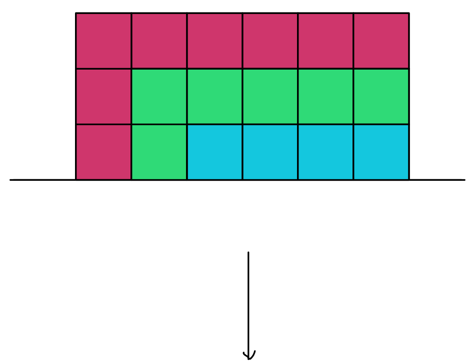
Starting with our previous rectangle…

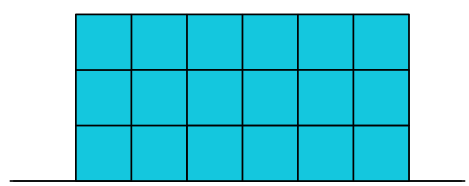


We add a new 'L' shape over it,



and the result is a new rectangle!





Let us define the following variables.

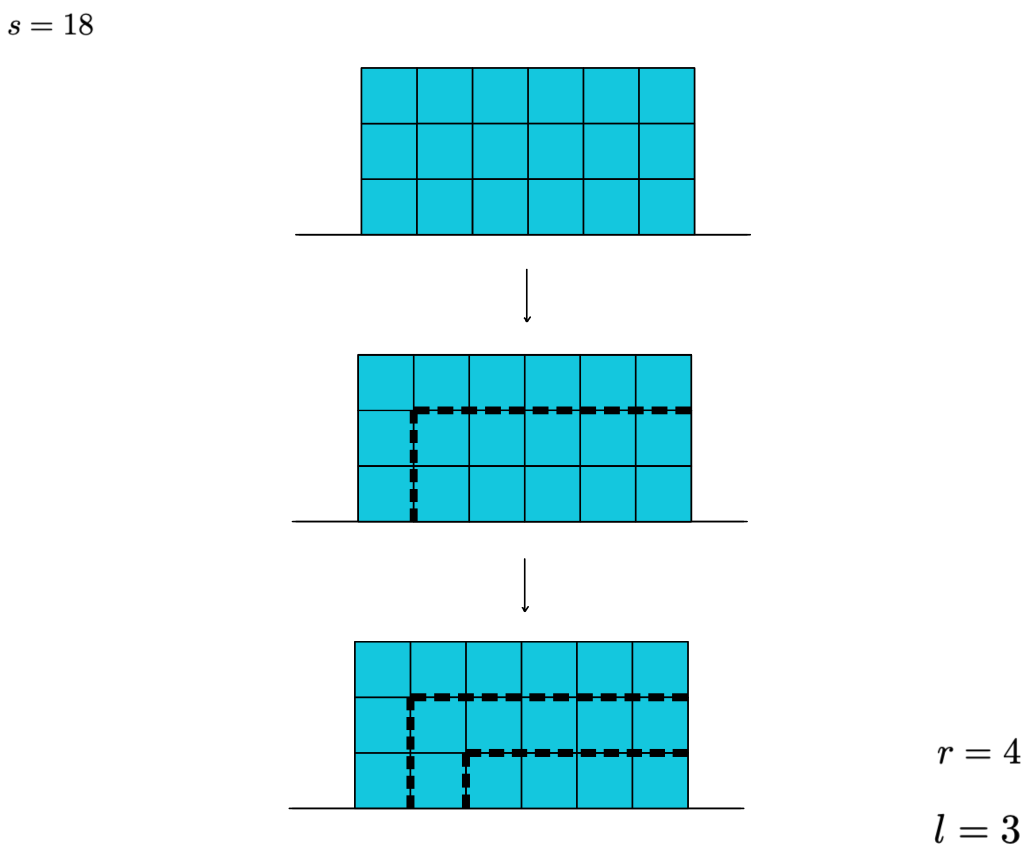
r = the number of blocks in the **starting row**.

l = the number of levels (or *layers*) of the long rectangle (at the beginning, this is always 1!)

s = the *total number of blocks in the rectangle at any stage*.

With these variables in mind, we may say that s is a **function** of r and l. In other words, given some specific pair (r, l), we can always create a long rectangle with a specific s number of blocks. We will describe this function as $s(r, l)$, where r and l are any natural numbers.

Before we move on, just know this: Given any long rectangle, you *can* visually identify r and l by shading in the layers, like this (using our previous example):



Recurrence Relations on Long Rectangles

With this idea of a long rectangle of $s(r, l)$ blocks in mind, here is a recurrence relation identity:

$$s(r,l) - s(r,l-1) = s(r-1,l) - s(r-1,l-1) + 1$$

Your first task is to verify this identity! In groups, make up your own long rectangles – you may choose any r and l you like (keep in mind that if either r or l become too large, your rectangle will also be pretty big!), and see if the identity checks out.

As you work, note any patterns you see.

What do you notice? Is the identity holding?

Part 2: A table for *s(r, l)*

Your next task is to use this identity to fill out the following table, where the columns are values of *l*, and the rows are values of *r*. (Hint: begin by generating the first row and column, using the rectangles!) Use the recurrence relation to fill out the rest of the table!

$$s\,(r,l) - s\,(r,l-1) = s\,(r-1,l) - s\,(r-1,l-1) + 1$$

$r \backslash l$	1	2	3	4	5	6	7	8
1								
2								
3								
4								
5								
6								
7								
8								

Part 3: Finding an explicit formula for s(*r, l*)

In your group, look at the table you created in the last step. Can you use it to try and find a *general formula* for *s* as a function of *r* and *l* (in other words, without using the recurrence relation)?

r \ l	1	2	3	4	5	6	7	8
1	1	4	9	16	25	36	49	64
2	2	6	12	20	30	42	56	72
3	3	8	15	24	35	48	63	80
4	4	10	18	28	40	54	70	88
5	5	12	21	32	45	60	77	96
6	6	14	24	36	50	66	84	104
7	7	16	27	40	55	72	91	112
8	8	18	30	44	60	78	98	120

Write down any patterns you see!

Part 4: Proving the Recurrence Relation

With this conjecture for $s(r, l)$ in mind, your final challenge is to *prove* the recurrence relation, using induction.

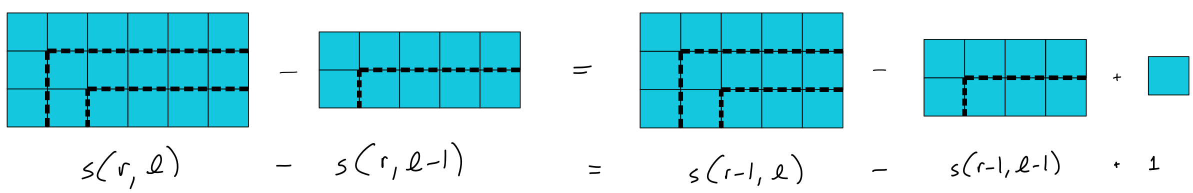
Answer Key

Recurrence Relations on Long Rectangles

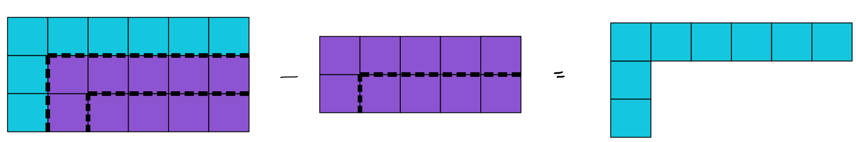
With this idea of a long rectangle of $s(r, l)$ blocks in mind, here is a recurrence relation identity:

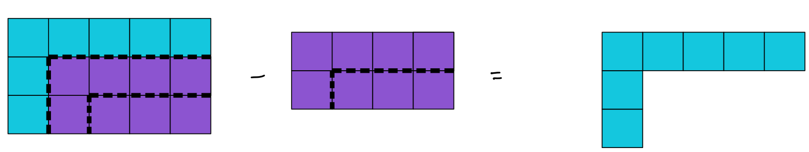
$$s(r,l) - s(r,l-1) = s(r-1,l) - s(r-1,l-1) + 1$$

A sample problem verifying the identity is given below, with $r = 4$ and $l = 3$.

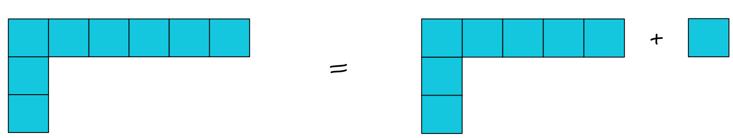


The intuitive/geometric connection to be made here is that on each side of the equal sign, the smaller long rectangle *nests* perfectly into the larger one, so that when each smaller rectangle is taken away, we are left with only the outermost layer:





It follows visually that the second of these outer layers is horizontally *one block shorter* than the first, as a result of it being constructed on a starting row of length $(r - 1)$. Thus, adding the single block back in completes the identity.



Part 4: Proving the Recurrence Relation

With this conjecture for *s*(*r, l*) in mind, your final challenge is to *prove* the recurrence relation, using induction.

<u>Proof</u>:

We wish to show, using induction, that

$$s\left(r,l\right)=l\cdot\left(l+r-1\right)$$

using the recurrence relation

$$s\left(r-1,l\right)+s\left(r,l-1\right)-s\left(r-1,l-1\right)+1=s\left(r,l\right)$$

Base case 1: $r = 1$.

$$s\left(1-1,l\right)+s\left(1,l-1\right)-s\left(1-1,l\right)+1$$

$$=l\left(l+\left(1-1\right)-1\right)+\left(l-1\right)\left(\left(l-1\right)+1-1\right)-\left(l-1\right)\left(\left(l-1\right)+\left(1-1\right)-1\right)+1$$

$$=l\left(l-1\right)+\left(l-1\right)\left(l-1\right)-\left(l-1\right)\left(l-2\right)+1$$

$$=l^2-l+l^2-2l+1-\left(l^2-3l+2\right)+1$$

$$=l^2-l+l^2-2l+1-l^2+3l-2+1$$

$$=l^2$$

$$=l\left(l+1-1\right)$$

$$=s\left(1,l\right)$$

Thus the first base case is satisfied.

Base case 2: $l = 1$.

$$s\left(r-1,1\right)+s\left(r,1-1\right)-s\left(r-1,1-1\right)+1$$

$$=1\cdot\left(1+\left(r-1\right)-1\right)+\left(1-1\right)\left(\left(1-1\right)+r-1\right)+\left(1-1\right)\left(\left(1-1\right)+\left(r-1\right)-1\right)+1$$

$$=1\cdot\left(r-1\right)+0\cdot\left(r-1\right)+0\cdot\left(r-2\right)+1$$

$$= r - 1 + 1$$

$$= r$$

$$= 1 \cdot (1 + r - 1)$$

$$= s(r, 1)$$

Thus the second base case is satisfied.

Inductive Hypothesis: For all natural numbers *i*, *j* such that $i \leq r, j \leq l, (i, j) \neq (r, l)$, assume that

$$s(i, j) = j \cdot (j + i - 1)$$

Using our recurrence relation identity, we have the following:

$$s(r, l) = s(r - 1, l) + s(r, l - 1) - s(r - 1, l - 1) + 1$$

We note that the argument of each of these sub-terms falls under our inductive hypothesis. Thus we have:

$$s(r - 1, l) + s(r, l - 1) - s(r - 1, l - 1) + 1$$

$$= l(l + (r - 1) - 1) + (l - 1)((l - 1) + r - 1) - (l - 1)((l - 1) + (r - 1) - 1) + 1$$

$$= l(l + r - 2) + (l - 1)(l + r - 2) - (l - 1)(l + r - 3) + 1$$

$$= l^2 + lr - 2l + \left(l^2 + lr - 2l - l - r + 2\right) - \left(l^2 + lr - 3l - l - r + 3\right) + 1$$

$$= l^2 + lr - 2l + l^2 + lr - 3l - r + 2 - l^2 - lr + 4l + r - 2$$

$$= l^2 + l - l$$

$$= l(l + r - 1)$$

$$= s(r, l)$$

And so, by induction, the proof is complete.

Reflections on the Activity

This activity has the potential to give students a new and unique way of learning to understand and prove recurrence relations from a geometric basis. Compared to the first activity, this one is less front-loaded, which I think could engage students more immediately. By keeping the emphasis less on function notation and more on the long rectangles themselves, this activity stays firmly in the visual realm / feels more like play, which could help students in tackling the more difficult topic of verifying recurrence relations. Having only spent a moderate amount of time with the underlying mathematics of this activity myself, I would suspect that there are even richer patterns hiding in it; off the top of my head, I can imagine expanding the 8x8 table, and also looking for recurrence relations buried in it that span nonadjacent squares. How would these make sense geometrically? I can't help but think that for each new recursive connection, there is an equivalently cool visual/geometric insight to be learned as well.

Activity 3: Using Conjugation on Integer Partitions

Introduction/Notes on the Activity

This activity, ideally, is intended to follow either of the previous activities – while students will find the idea of conjugate partitions a fairly straightforward one to grasp, having been exposed to the Long Rectangles partitions will set students up well for some of the techniques found in this activity.

The focus of this activity is to allow students to explore the concept of *self-conjugate partitions* from a few different angles. To begin, students are shown examples of what conjugation means, and how to view it *visually* (using Ferrers diagrams); from there, they are introduced to the concept of self-conjugate partitions as a 'special case' of conjugation partitions.

The activity is broken into three parts. The first acts as a warm-up, giving students certain self-conjugate partitions (and modeling the various overall ways they can appear) while giving them practice at transcribing the Ferrers diagrams into sums of natural numbers. The second part provides students with some natural number *n* and asks them to use Ferrers diagrams (or other visual representations) to try and *construct* as many self-conjugate partitions of that *n* as possible on their own. This activity is to have a minimum of scaffolding; by the nature of self-conjugate partitions, students should be encouraged to explore the topic without too much direction. If the instructor feels it appropriate, students may be encouraged to consider the extreme case of the "long L" self-conjugate partition.

The final part of the activity approaches the construction of self-conjugate partitions from a different angle – using *perfect square partitions*. The idea is that by this point in the activity students will have gotten familiar with the diagonal symmetry that marks all self-conjugate partition; so, as an opposite extreme case, any perfect square partition (i.e. a partition where all parts are the same size) is trivially self-conjugate. Students will be given a sample perfect square partition color-coded such that they see it as *several* nested squares (this visual will stand in contrast to the ‘outer L’ technique used in the Long Rectangles activity). Having just attempted to find self-conjugate partitions from a different perspective, students will then get to play with removing parts of the different outer layers (or removing entire layers) to see what they can do.

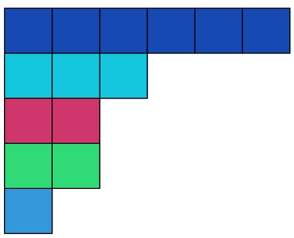
This activity, even if it is not explicitly emphasized in the worksheet, contains rich connections to integer partition patterns, relationships to core partition identities (such as there being a bijection between self-conjugate partitions and distinct odd partitions), and number theory. It is meant to be just a first taste of this rich topic for students, but the instructor may wish to emphasize or further develop any of these threads as they see fit.

Worksheet

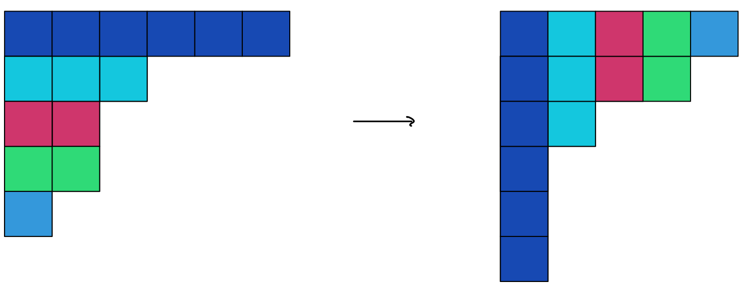
<u>Conjugations on Integer Partitions</u>

In this activity, we're going to be explore some neat properties of integer partitions using a transformation called *conjugation*. Here is how it works.

Suppose we are given the following partition of $n = 14$: $6 + 3 + 2 + 2 + 1$



Conjugation would take this partition and *transpose its columns and rows,* as shown below (colors are used to show the transformation).



This takes the partition $6 + 3 + 2 + 2 + 1$ and turns it into a new partition: $5 + 4 + 2 + 1 + 1 + 1$.

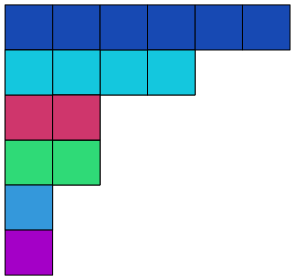
We say that these are *conjugate partitions*, since under conjugation, each is transformed into the other.

after this example:

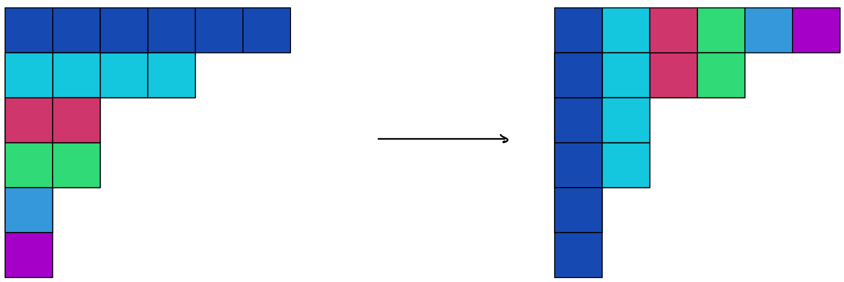
Self-Conjugate Partitions

In this activity, we're going to focus in particular on something called *self-conjugate* partitions – partitions which are their own conjugate.

For example: Here is a partition of $n = 14$:



Under conjugation, we see that the partition *does not change*:

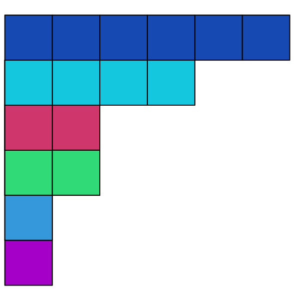


This is what we mean by a *self-conjugate partition*.

Part 1: Transcribing Self-Conjugate Partitions

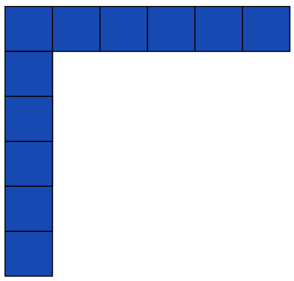
To warm up to the main activity, we will begin by looking at some self-conjugate partitions. Based on your knowledge of integer partitions and Ferrers diagrams, your task will be to describe each partition *numerically* – in other words, to write each n as the sum of the parts.

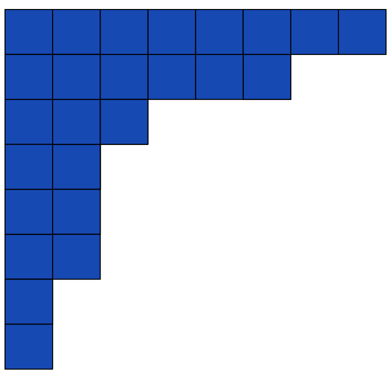
For example, the above self-conjugate partition of $n = 14$ would be:

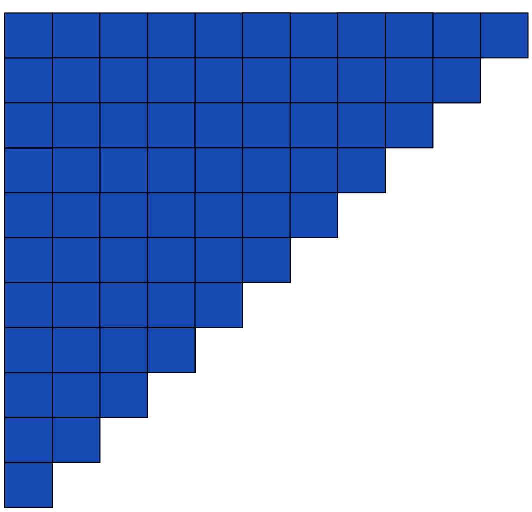


$$14 = 6 + 4 + 2 + 2 + 1 + 1$$

Several self-conjugate partitions (on various values of n) are given below. Describe each numerically, and note any patterns you see.







What do you notice?

Part 2: Constructing Self-Conjugate Partitions

In this activity, each group will be given a different value for n. Your task is to find *as many self-conjugate partitions of this number as possible.*

You are encouraged to draw pictures to help constructing these partitions. As you work, keep the following questions in mind:

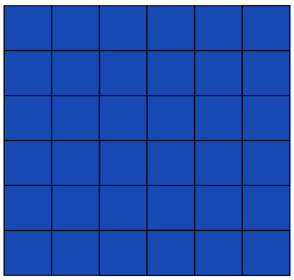
- What patterns do you notice?
- Are there any strategies you found to make the process easier?
- How might you know when you've found *all* the possible self-conjugate partitions?

Your number is n = _________.

Part 3: Using Perfect Squares to Construct Self-Conjugate Partitions

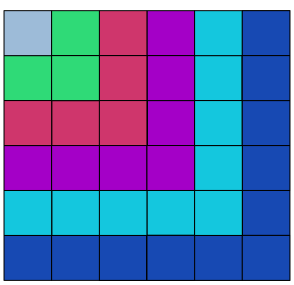
Based on the previous activity, you may have noticed that in order to create different self-conjugate partitions on a fixed n, there are certain constraints on how to take away blocks from the 'outer shell' of the partition, and replace them elsewhere.

To look at this from another angle, we now consider *perfect square partitions*. For example, if $n = 36$,



is a perfect square partition (in other words, a partition where all parts are the same size).

Let us look at this partition using colors:



With this multicolor representation, do you agree that this perfect square partition is also *self-conjugate*? Why or why not?

If you believe this, how could you use this "square representation" to construct other self-conjugate partitions of $n = 36$ by removing some (or all) of each outer layer of the partition?

What do you notice?

Can you apply any techniques you found here to find self-conjugate partitions on perfect square partitions for larger numbers?

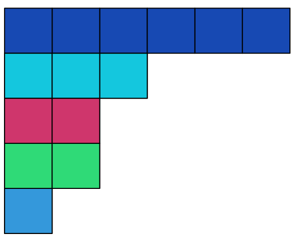
Homework: Play this out on larger perfect squares. Challenge yourself! Try $n = 64$, $n = 81$, or even higher!

Worksheet Answer Key

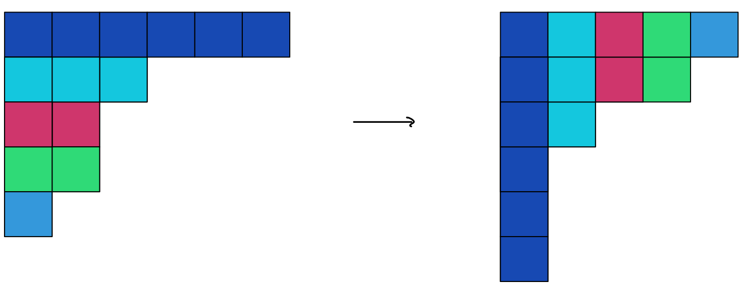
<u>Conjugations on Integer Partitions</u>

In this activity, we're going to be explore some neat properties of integer partitions using a transformation called *conjugation*. Here is how it works.

Suppose we are given the following partition of $n = 14$: $6 + 3 + 2 + 2 + 1$



Conjugation would take this partition and *transpose its columns and rows,* as shown below (colors are used to show the transformation).



This takes the partition $6 + 3 + 2 + 2 + 1$ and turns it into a new partition: $5 + 4 + 2 + 1 + 1 + 1$.

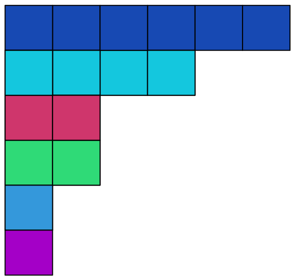
We say that these are *conjugate partitions*, since under conjugation, each is transformed into the other.

after this example:

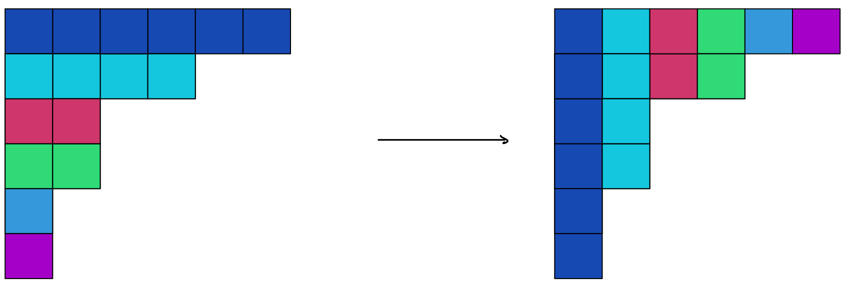
Self-Conjugate Partitions

In this activity, we're going to focus in particular on something called *self-conjugate* partitions – partitions which are their own conjugate.

For example: Here is a partition of $n = 14$:



Under conjugation, we see that the partition *does not change*:

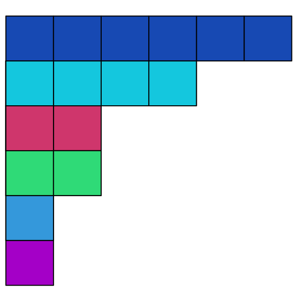


This is what we mean by a *self-conjugate partition*.

Part 1: Transcribing Self-Conjugate Partitions

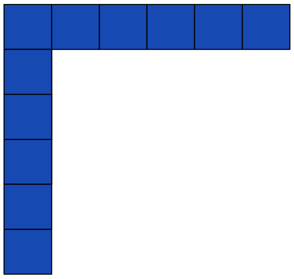
To warm up to the main activity, we will begin by looking at some self-conjugate partitions. Based on your knowledge of integer partitions and Ferrers diagrams, your task will be to describe each partition *numerically* – in other words, to write each n as the sum of the parts.

For example, the above self-conjugate partition of $n = 14$ would be:



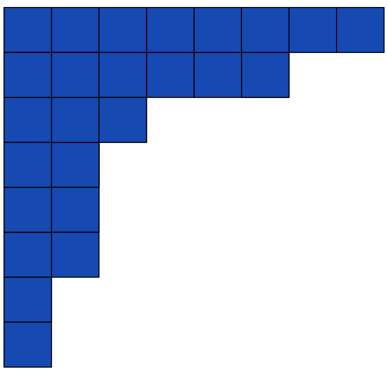
$$14 = 6 + 4 + 2 + 2 + 1 + 1$$

Several self-conjugate partitions (on various values of n) are given below. Describe each numerically, and note any patterns you see.

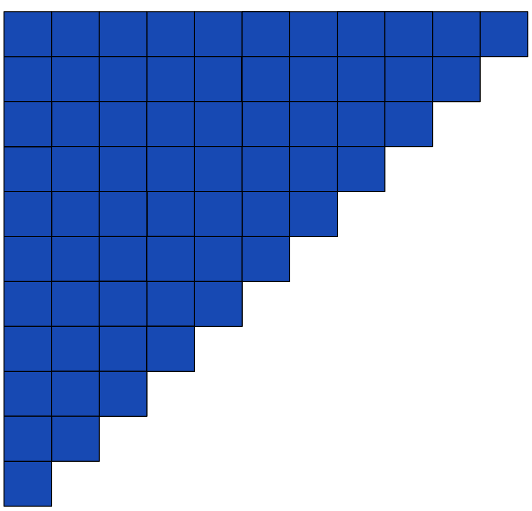


$11 = 6 + 1 + 1 + 1 + 1 + 1$

In general, the main observation students may see is that the *largest* part corresponds to the total *number* of parts.



$25 = 8 + 6 + 3 + 2 + 2 + 2 + 1 + 1$



$66 = 11 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$

In this case, students may note that the nature of this particular partition (each part differing by 1 from the previous part) lends itself to a familiar summation identity

$$\sum_{n=1}^{11} = \frac{11\,(11+1)}{2}$$

$$= 66$$

What do you notice?

Part 2: Constructing Self-Conjugate Partitions

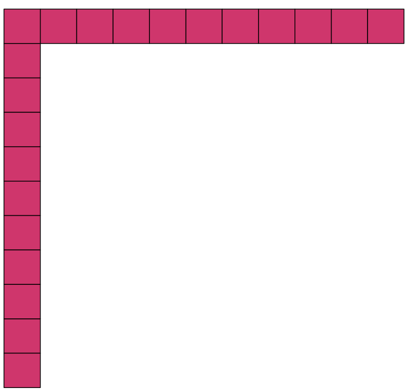
In this activity, each group will be given a different value for n. Your task is to find *as many self-conjugate partitions of this number as possible.*

You are encouraged to draw pictures to help constructing these partitions. As you work, keep the following questions in mind:

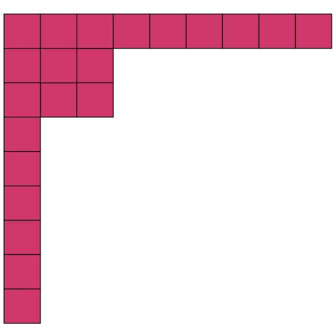
- What patterns do you notice?
- Are there any strategies you found to make the process easier?
- How might you know when you've found *all* the possible self-conjugate partitions?

Your number is $n =$ ___21____.

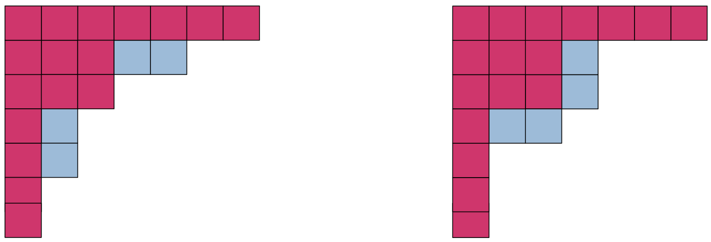
Using $n = 21$ as an example, some groups may hit upon the technique of trying to begin from the 'extreme' case of creating a self-conjugate partition as a *long L*:



From this extreme case, students may proceed by removing (symmetrically) blocks from the 'ends' of the L. Each pair of end-blocks removed will give two blocks to place elsewhere; however, hey may notice that in order to fill in any of the interior of the partition while still keeping it self-conjugation, they will only be able to use *an even number of blocks that can be split into odd piles.* For example, by removing two blocks from each end of the L, the following partition may be constructed:



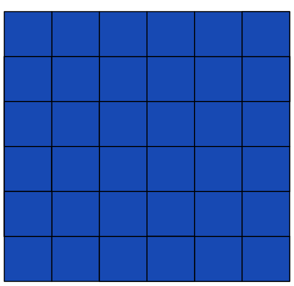
and so on. However, the more students dig into this activity, the more they will start to have different options in terms of *where* to remove certain blocks in order to free up/create other self-conjugate partitions. To give one more example on the above partition, if we remove two blocks from each end of the L, we have at least two options for how to replace them that preserves the symmetry:



Part 3: Using Perfect Squares to Construct Self-Conjugate Partitions

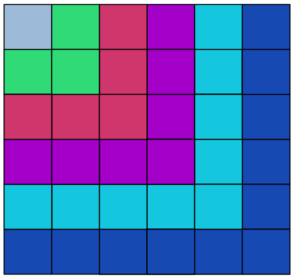
Based on the previous activity, you may have noticed that in order to create different self-conjugate partitions on a fixed n, there are certain constraints on how to take away blocks from the 'outer shell' of the partition, and replace them elsewhere.

To look at this from another angle, we now consider *perfect square partitions*. For example, if $n = 36$,



is a perfect square partition (in other words, a partition where all parts are the same size).

Let us look at this partition using colors:



With this multicolor representation, do you agree that this perfect square partition is also *self-conjugate*? Why or why not?

By this point in the activity, students will be attuned to the *diagonal symmetry* that marks all self-conjugate partitions.

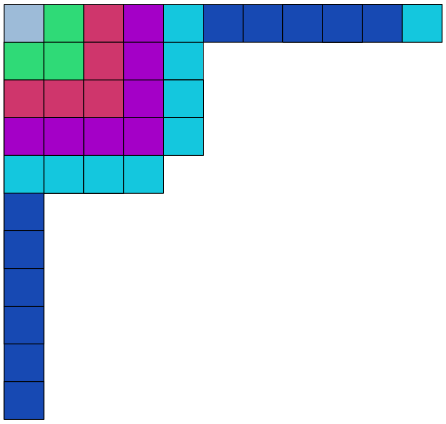
If you believe this, how could you use this "square representation" to construct other self-conjugate partitions of $n = 36$ by removing some (or all) of each outer layer of the partition?

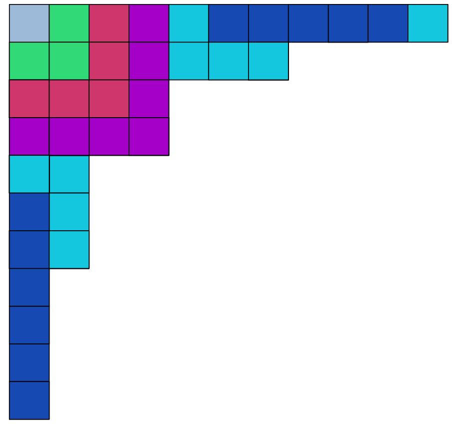
What do you notice?

Can you apply any techniques you found here to find self-conjugate partitions on perfect square partitions for larger numbers?

The intent with this part of the activity is to encourage students to look at manipulating their partitions from the other extremal case (in other words, the counterpoint to the long L). By construction, a perfect square partition is self-conjugate; by see it as *several* squares nested together, students are visually led towards techniques that peel away outer layer by outer layer, while never losing the foundation of some "core" of inner perfect squares. This hints at the idea of **Durfee squares** being central to self-conjugate partitions, without needing to explicitly broach the subject in this activity (this may be worth considering as an extension, though!).

So, some examples of student work might be:





and so on. By keeping the multicolor square layers present, students will have a nice foundation to more freely play with different self-conjugate partitions, and ideally feel more confident in their constructions than in the previous section.

Reflections on the Activity

This activity arose out of a struggle to find a solid way to introduce students to the idea of conjugation on partitions; originally, I'd planned on constructing the activity around the bijective identity between self-conjugate partitions and distinct odd partitions; however, in trying to develop that, it became clear that as interesting as that is as a bijection, it's fairly short/lacking in terms of material to base an activity upon (the visual "method" for verifying it can be shown quickly). I'd originally conceived of providing students with entire sets of partition numbers, and then having them identify all self-conjugate and distinct odd partitions within the set, with the ultimate goal of leading them to this identity by the number of each being the same. But this was a fairly flimsy pretense to build an activity upon, and it also became apparent that even doing a few examples of these countings would not necessarily lead students to arrive at the desired identity (and more importantly, not give them any insight into *why* it was true). So this was abandoned in favor of the above activity instead.

In its current version, my feeling is that this activity, paired with either of the previous Long Rectangle activities, will give students a solid introduction to some of the fascinating mathematics of integer partitions, and connect to each other well enough to form a basis for future study. As mentioned in the beginning of this paper, the activities and topics focused on here barely scratch the surface of the depth and breadth of the world of integer partitions; many more curricula could be developed on topics such as generating functions, Ramanujan identities, and more.

Extensions

As mentioned above, one thing implicit in the third part of this activity is the idea of *Durfee squares*; this activity may be extended, if the instructor wishes, to make this connection more explicit (and to use it as a lead-in to any number of other activities which Durfee squares connect to).

Another way that this activity could extend is by stepping *away* from the world of self-conjugate partitions into the larger pool of conjugate partitions; by removing the condition of being self-conjugate (but now having some experience and practice with them), students can analyze and compare the patterns they see by studying conjugate partitions more generally, which may lead towards some powerful bijective identities.

Also buried in this activity was the connection between self-conjugate partitions and partitions into distinct odd parts. Even though the verification of that bijection might not be enough for its own activity, it may act as a bridge to activities which focus on identities involving distinct parts. One such identity, which is not at all obvious, is:

$$p(\, n \mid distinct\ parts) = p(\, n \mid parts\ of\ \ every\ size\ from\ 1\ to\ the\ largest\ part)$$

This is just one direction that exploration of distinct identities could go.

Finally, this activity contains subtle hints of the significance of the parts of each parts being *odd or even*. One extension in this direction would be an an activity asking students to *remove* odd partitions from the set of *all* partitions for some given *n*, and ask them what sorts of partitions remain (in this case, the set of all partitions of either even parts, or a mixture of even and odd parts).

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